Thus $T_x(Z)$ is a subspace of the kernel that has the same dimension as the complete kernel; hence $T_x(Z)$ must be the kernel. Q.E.D.

EXERCISES

*1. If $f: X \longrightarrow Y$ is a submersion and U is an open set of X, show that $f(U)$ is open in Y.

*2. (a) If X is compact and Y connected, show every submersion $f: X \longrightarrow Y$ is surjective.
(b) Show that there exist no submersions of compact manifolds into Euclidean spaces.

3. Show that the curve $t \longrightarrow (t, t^2, t^3)$ embeds $\mathbf{R}^1$ into $\mathbf{R}^3$. Find two independent functions that globally define the image. Are your functions independent on all of $\mathbf{R}^3$, or just on an open neighborhood of the image?

4. Prove the following extension of Partial Converse 2. Suppose that $Z \subset X \subset Y$ are manifolds, and $z \in Z$. Then there exist independent functions $g_1, \ldots, g_l$ on a neighborhood W of z in Y such that

$$Z \cap W = \{y \in W : g_1(y) = 0, \ldots, g_l(y) = 0\}$$

and

$$X \cap W = \{y \in W : g_1(y) = 0, \ldots, g_m(y) = 0\},$$

where $l - m$ is the codimension of Z in X.

5. Check that 0 is the only critical value of the map $f: \mathbf{R}^3 \longrightarrow \mathbf{R}^1$ defined by

$$f(x, y, z) = x^2 + y^2 - z^2.$$

Prove that if a and b are either both positive or both negative, then $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic. [HINT: Consider scalar multiplication by $\sqrt{b/a}$ on $\mathbf{R}^3$.] Pictorially examine the catastrophic change in the topology of $f^{-1}(c)$ as c passes through the critical value.

6. More generally, let p be any homogeneous polynomial in k-variables. Homogeneity means

$$p(tx_1, \ldots, tx_k) = t^m p(x_1, \ldots, x_k).$$

Prove that the set of points x, where $p(x) = a$, is a $k - 1$ dimensional submanifold of $\mathbf{R}^k$, provided that $a \neq 0$. Show that the manifolds obtained with $a > 0$ are all diffeomorphic, as are those with $a < 0$. [HINT:

Use Euler's identity for homogeneous polynomials

$$\sum_{i=1}^{k} x_i \frac{\partial p}{\partial x_i} = \mathrm{m} \cdot p$$

to prove that 0 is the only critical value of p.]

*7. (*Stack of Records Theorem.*) Suppose that y is a regular value of $f: X \longrightarrow Y$, where X is compact and has the same dimension as Y. Show that $f^{-1}(y)$ is a finite set $\{x_1, \dots, x_N\}$. Prove there exists a neighborhood U of y in Y such that $f^{-1}(U)$ is a disjoint union $V_1 \cup \cdots \cup V_N$, where V_i is an open neighborhood of x_i and f maps each V_i diffeomorphically onto U. [HINT: Pick disjoint neighborhoods W_i of x_i that are mapped diffeomorphically. Show that $f(X - \cup W_i)$ is compact and does not contain y.] See Figure 1-13.

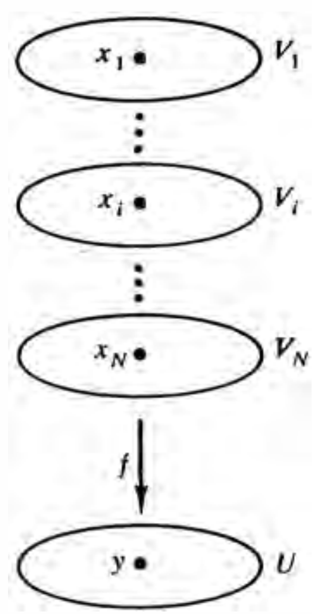

Figure 1-13

8. Let

$$p(z) = z^m + a_1 z^{m-1} + \cdots + a_m$$

be a polynomial with complex coefficients, and consider the associated map $z \dashrightarrow p(z)$ of the complex plane $\mathbf{C} \longrightarrow \mathbf{C}$. Prove that this is a submersion except at finitely many points.

9. Show that the orthogonal group $O(n)$ is compact. [HINT: Show that if $A = (a_{ij})$ is orthogonal, then for each i, $\sum_j a_{ij}^2 = 1$.]

10. Verify that the tangent space to $O(n)$ at the identity matrix I is the vector space of skew symmetric $n \times n$ matrices—that is, matrices A satisfying $A^t = -A$.

11. (a) The $n \times n$ matrices with determinant $+1$ form a group denoted $SL(n)$. Prove that $SL(n)$ is a submanifold of $M(n)$ and thus is a Lie group. [HINT: Prove that 0 is the only critical value of det: $M(n) \longrightarrow R$. In fact, if det $(A) \neq 0$, then show that det is already a submersion when restricted to the set $\{tA, t > 0\}$. Remark: This is really a special case of Exercise 5.]

(b) Check that the tangent space to $SL(n)$ at the identity matrix consists of all matrices with trace equal to zero.

12. Prove that the set of all 2×2 matrices of rank 1 is a three-dimensional submanifold of $\mathbf{R}^4 = M(2)$. [HINT: Show that the determinant function is a submersion on the manifold of nonzero 2×2 matrices $M(2) - \{0\}$.]

13. Prove that the set of $m \times n$ matrices of rank r is a submanifold of $\mathbf{R}^{mn}$ of of codimension $(m - r)(n - r)$. [HINT: Suppose, for simplicity, that an $m \times n$ matrix A has the form

$$A = \begin{array}{c} \\ r \\ m-r \end{array}\!\!\begin{array}{c} \begin{array}{cc} r & n-r \end{array} \\ \left(\begin{array}{c|c} B & C \\ \hline D & E \end{array}\right) \end{array},$$

where the $r \times r$ matrix B is nonsingular. Postmultiply by the nonsingular matrix

$$\left(\begin{array}{c|c} I & -B^{-1}C \\ \hline 0 & I \end{array}\right)$$

to prove that rank $(A) = r$ if and only if $E - DB^{-1}C = 0$.]

§5 Transversality

We have observed that the solutions of an equation $f(x) = y$ form a smooth manifold, provided that y is a regular value of the map $f: X \longrightarrow Y$. Consider, now, sets of points in X whose functional values are constrained, not necessarily to be a constant y, but to satisfy an arbitrary smooth condition. Thus assume Z to be a submanifold of Y, and examine the set of solutions of the relation $f(x) \in Z$. When can we be assured that this solution set, the preimage $f^{-1}(Z)$, is a tractable geometric object? This question will lead us to define a new differential property, an extension of the notion of regularity, which will become the major theme of the book.